

RELAXATION PATTERNS AND SEMI-MARKOV DYNAMICS

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ABSTRACT. Exponential relaxation to equilibrium is a typical property of physical systems, but inhomogeneities are known to distort the exponential relaxation curve, leading to a wide variety of relaxation patterns. Power law relaxation is related to fractional derivatives in the time variable. More general relaxation patterns are considered here, and the corresponding semi-Markov processes are studied. Our method, based on Bernstein functions, unifies three different approaches in the literature.

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1. INTRODUCTION

Relaxation phenomena in complex systems can deviate from the traditional exponential model. In a heterogeneous system, a linear combination of exponential curves with varying rates can lead to power law relaxation, or a variety of other forms. Power law (Cole-Cole) relaxation and Havriliak-Negami relaxation (transitioning between power laws frequency changes) are commonly seen in complex materials, including polymers, disordered crystals, supercooled liquids, and amorphous semiconductors [20; 23; 48]. The connection between relaxation and continuous time random walk (CTRW) models is reviewed in [54]. In the CTRW model, particle motions X_n are separated by random waiting times W_n , and the long-time limiting particle density solves an evolution equation that incorporates the relaxation curve. One famous example is the fractional Fokker-Planck equation for subdiffusive particle motions in a potential well, where delays in particle motion caused by sticking or trapping with a power-law distributed waiting time

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lead to a fractional time derivative in the evolution equation for particle density [21; 29; 30; 41; 50]. More general waiting time distributions lead to a variety of pseudo-differential operators in time [35] that model general relaxation patterns.

This paper applies the theory of Bernstein functions to unify the three main approaches to relaxation modeling, exemplified by the work of Meerschaert and Scheffler [35], Toaldo [51], and Magdziarz and Schilling [31]. We show that all three approaches are equivalent, and we establish the correspondence between model equations using conjugate Bernstein functions [49]. We establish some properties of solutions using regular variation theory [12; 18], and extend to semi-Markov (CTRW) particle models. The forward and backward Kolmogorov equations for Markov processes on a discrete state space are generalized to the semi-Markov case, and the classification of states into transient or persistent is discussed. More general evolution equations, solved by time-changed (relaxed) semigroups on a Hilbert space, are also considered.

2. RELAXATION PATTERNS

2.1. Some basic facts on Bernstein functions and subordinators. In order to go on with the results we need some theory of Bernstein functions and subordinators. We recall here some basic facts (see Bertoin [9, 10]; Schilling et al. [49] for more details). A Bernstein function $f : (0, \infty) \rightarrow [0, \infty)$ is defined to be of class C^∞ and such that $(-1)^{n-1} f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N} := \{1, 2, 3, \dots\}$. A function f is a Bernstein function if and only if [49, Thm 3.2] it can be written in the form

$$f(\phi) = a + b\phi + \int_0^\infty (1 - e^{-\phi s}) \nu(ds) \quad (2.1)$$

where a and b are non-negative constants and $\nu(\cdot)$ is a σ -finite measure on $(0, \infty)$ such that the integrability condition

$$\int_0^\infty (s \wedge 1) \nu(ds) < \infty \quad (2.2)$$

is fulfilled. The triplet (a, b, ν) is said to be a Lévy triplet. An integration by parts of (2.1) yields

$$\phi^{-1} f(\phi) = b + \int_0^\infty e^{-\phi s} \bar{\nu}(s) ds \quad (2.3)$$

where $\bar{\nu}(s) = a + \nu(s, \infty)$. It follows from [49, Corollary 3.7 (iv)] that (2.3) is a completely monotone function i.e. it is C^∞ and such that

$$(-1)^n \frac{d^n}{d\phi^n} (\phi^{-1} f(\phi)) \geq 0, \text{ for all } n \in \mathbb{N}. \quad (2.4)$$

A particular subset of Bernstein functions is the set of special Bernstein functions. A Bernstein function f is said to be special if $f^*(\phi) = \phi/f(\phi)$ is again a Bernstein function. The function f^* , which is also special, is said to be the conjugate of f and has the representation

$$f^*(\phi) = \frac{\phi}{f(\phi)} = a^* + b^* \phi + \int_0^\infty (1 - e^{-\phi s}) \nu^*(ds) \quad (2.5)$$

where [49, p. 93]

$$b^* = \begin{cases} 0, & b > 0, \\ \frac{1}{a+\nu(0,\infty)}, & b = 0, \end{cases}, \quad a^* = \begin{cases} 0, & a > 0, \\ \frac{1}{b+\int_0^\infty t\nu(dt)}, & a = 0. \end{cases} \quad (2.6)$$

In what follows we will also need complete Bernstein functions. A Bernstein function f is said to be complete [49, Def 6.1] if the density $\nu(s)$ of the Lévy measure $\nu(ds) = \nu(s) ds$ appearing in (2.1) is a completely monotone function. According to [49, Thm 6.2, (ii)] we have that a Bernstein function f is complete if and only if $\phi \rightarrow \phi^{-1}f(\phi)$ is a (non-negative) Stieltjes function i.e. a function $h : (0, \infty) \rightarrow [0, \infty)$ which can be written in the form

$$h(\phi) = \frac{a}{\phi} + b + \int_0^\infty \frac{1}{\phi + s} \mathfrak{s}(ds) \quad (2.7)$$

where \mathfrak{s} is a measure on $(0, \infty)$ such that

$$\int_0^\infty (1+s)^{-1} \mathfrak{s}(ds) < \infty. \quad (2.8)$$

It is also true that f is complete if and only if the conjugate $f^*(\phi) = \phi/f(\phi)$ is complete, and hence every complete Bernstein function is special [49, Prop 7.1].

Bernstein functions are naturally associated with subordinators which are non-decreasing Lévy processes. The one-dimensional distributions of a subordinator form a convolution semigroup of sub-probability measures, i.e., a family of sub-probability measures $\{\mu_t(\cdot)\}_{t \geq 0}$ supported on $(0, \infty)$ such that

- (1) $\mu_t(0, \infty) \leq 1$,
- (2) $\mu_t * \mu_s = \mu_{t+s}$ for all $s, t \geq 0$,
- (3) $\mu_t \rightarrow \delta_0$ vaguely as $t \rightarrow 0$,

and

$$\mathcal{L}[\mu_t(\cdot)](\phi) = e^{-tf(\phi)} \quad (2.9)$$

where $f(\phi)$ is a Bernstein function. We will denote by $\sigma^f(t)$, $t \geq 0$, the subordinator with Laplace exponent f . If f is a special Bernstein function then the corresponding subordinator is also said to be special. The following facts will be used throughout the paper. A subordinator is special if and only if [49, Thm 10.3] its potential measure

$$U^{\sigma^f}(dt) := \mathbb{E} \int_0^\infty \mathbf{1}_{\{\sigma^f(s) \in dt\}} ds = c\delta_0(dt) + u_{\sigma^f}(t)dt \quad (2.10)$$

for some $c \geq 0$ and some non-increasing function $u_{\sigma^f} : (0, \infty) \rightarrow (0, \infty)$ satisfying $\int_0^1 u(t)dt < \infty$. In particular from [49, Corollary 10.8] we know that if $b = 0$ and $\nu(0, \infty) = \infty$ then $c = b^* = 0$, $\nu^*(0, \infty) = \infty$, and

$$u_{\sigma^f}(t) = a^* + \nu^*(t, \infty) = \bar{\nu}^*(t). \quad (2.11)$$

2.2. Relaxation patterns. Meerschaert and Scheffler [35] develop limit theory for the continuous time random walk (CTRW) model from statistical physics. Given an iid sequence of jumps J_n and an iid sequence of waiting times W_n , a particle jumps to location $S_n = J_1 + \dots + J_n$ at time $T_n = W_1 + \dots + W_n$. Given a convergent triangular array of CTRW models, they show [35, Theorem 2.1] that

the limit process is of the form $X(L^f(t))$ where $X(t)$ is the limiting Lévy process for the random walk of jumps, time changed by the inverse subordinator

$$L^f(t) = \inf \{s \geq 0 : \sigma^f(s) > t\}, \quad (2.12)$$

where $\sigma^f(t)$ is a Lévy subordinator with Laplace exponent f , so that $\mathbb{E}[e^{-\phi\sigma^f(t)}] = e^{-tf(\phi)}$ for all $t \geq 0$. Kolokoltsov [27] extended the model to a Markov process limit $X(t)$ by allowing the distribution of the jumps J_n to vary in space. In both cases (under some mild conditions, see [35, Theorem 4.1] and [27, Theorem 4.2]) the probability densities $p(x, t)$ of the CTRW limit solve a governing equation

$$\mathbb{C}_f(\partial_t)p(x, t) = Ap(x, t) \quad (2.13)$$

where A is the generator of the Markov semigroup, and the Caputo-like operator

$$\mathbb{C}_f(\partial_t)u(t) = \mathcal{L}^{-1} [f(\phi)\tilde{u}(\phi) - \phi^{-1}f(\phi)u(0)](t), \quad (2.14)$$

see [35, Remark 4.8]. If the waiting times W_n belong to the domain of attraction of a stable subordinator with Laplace exponent $f(\phi) = \phi^\beta$, then (2.13) specializes to

$$\partial_t^\beta p(x, t) = Ap(x, t) \quad (2.15)$$

where ∂_t^β is the Caputo fractional derivative [37, Eq. (2.16)]. Other choices of f lead to distributed order [34] and tempered [5; 13] fractional derivatives.

Toaldo [51, Eq. (2.18)] introduced the operator

$$\mathcal{D}^f u(t) = b \frac{d}{dt} u(t) + \frac{d}{dt} \int_0^t u(s) \bar{\nu}(t-s) ds \quad (2.16)$$

where $\bar{\nu}(s) = a + \nu(s, \infty)$ for a Lévy triplet (a, b, ν) , and where $s \rightarrow \bar{\nu}(s)$ is assumed to be absolutely continuous on $(0, \infty)$. The operator (2.16) can be regularized by subtracting an “initial condition”, as in [35, Remark 4.8], resulting in a generalized Dzerbayshan-Caputo derivative

$$\mathfrak{D}_t^f u(t) = b \frac{d}{dt} u(t) + \frac{d}{dt} \int_0^t u(s) \bar{\nu}(t-s) ds - \bar{\nu}(t)u(0). \quad (2.17)$$

Use (2.3) to compute the Laplace symbol of (2.17) as

$$\begin{aligned} \mathcal{L} \left[\mathfrak{D}_t^f u(t) \right] (\phi) &= b\phi\tilde{u}(\phi) - bu(0) + \phi \mathcal{L} [u * \bar{\nu}] (\phi) - \left(\frac{f(\phi)}{\phi} - b \right) u(0) \\ &= b\phi\tilde{u}(\phi) - bu(0) + \phi\tilde{u}(\phi) \left(\frac{f(\phi)}{\phi} - b \right) - \left(\frac{f(\phi)}{\phi} - b \right) u(0) \\ &= f(\phi)\tilde{u}(\phi) - \phi^{-1}f(\phi)u(0). \end{aligned} \quad (2.18)$$

This shows that (2.14) and (2.17) are the same operator at least for continuously differentiable functions u : indeed the Laplace transforms agree and $t \rightarrow \mathfrak{D}_t^f u(t)$ is a continuous function since by [51, Proposition 2.7] we can write

$$\begin{aligned} \mathfrak{D}_t^f u(t) &= b \frac{d}{dt} u(t) + \frac{d}{dt} \int_0^t u(s) \bar{\nu}(t-s) ds - \bar{\nu}(t)u(0) \\ &= b \frac{d}{dt} u(t) + \int_0^t u'(s) \bar{\nu}(t-s) ds \end{aligned} \quad (2.19)$$

and therefore $\mathfrak{D}_t^f u(t)$ is continuous, since u' and $\bar{\nu}$ are continuous, hence also $u' * \bar{\nu}$. Hence (2.17) provides an explicit definition of the operator \mathbb{C}_f in (2.13).

A third approach was adopted in Magdziarz and Schilling [31]: The authors pointed out that the distribution (one-dimensional marginal) of $B(L^f(t))$, a special case of the CTRW scaling limit where B is a Brownian motion, is the fundamental solution to the generalized diffusion equation

$$\frac{\partial}{\partial t} q(x, t) = \frac{1}{2} \Phi_t \frac{\partial^2}{\partial x^2} q(x, t), \quad x \in \mathbb{R}, t > 0, \quad (2.20)$$

where Φ_t is the integro-differential operator

$$\Phi_t u(t) = \frac{d}{dt} \int_0^t u(s) M(t-s) ds \quad (2.21)$$

for a kernel $M(t)$ such that

$$\mathcal{L}[M(t)](\phi) = \frac{1}{f(\phi)}. \quad (2.22)$$

A special case of (2.20) called the fractional Fokker-Planck equation, with $M(s) = s^{-\alpha}/\Gamma(1-\alpha)$ for some $0 < \alpha < 1$, was introduced by Metzler et al. [41] in the physics literature, see also Henry et al. [21]. The extension to a general waiting time distribution, and hence a general time-convolution operator Φ_t , was pioneered by Sokolov and J. Klafter [50] in the context of statistical physics, see also Magdziarz [30], and in the mathematical literature by Magdziarz [29]. This form (2.20) of the CTRW limit equation is needed when one wants to add a source/sink term with the natural units of x/t , see Baeumer et al. [4] for additional discussion.

In this work we place these different approaches in a unifying framework by appealing to the theory of special Bernstein functions. Let f be the special Bernstein function (2.1) with conjugate (2.5) where a^* and b^* are given by (2.6). Assume that $\nu(0, \infty) = \infty$, so that $b^* = 0$ in view of (2.6). As in (2.3), an integration by parts in (2.5) yields

$$\phi^{-1} f^*(\phi) = b^* + \int_0^\infty e^{-\phi s} \bar{\nu}^*(s) ds \quad (2.23)$$

which implies that

$$\mathcal{L}[\bar{\nu}^*(t)](\phi) = \phi^{-1} f^*(\phi) = \frac{1}{f(\phi)} \quad (2.24)$$

since $f^*(\phi) = \phi/f(\phi)$. Then we may write $\Phi_t u(t) = \frac{d}{dt} \Psi_t u(t)$ where the operator

$$\Psi_t u(t) = \int_0^t u(s) M(t-s) ds, \quad (2.25)$$

and $\mathcal{L}[\Psi_t u(t)] = f(\phi)^{-1} \tilde{u}(\phi)$ for continuously differentiable functions u . Hence (2.21) is related to the conjugate Bernstein function f^* , while (2.14) and (2.16) are related to the Bernstein function f . In particular, we have $\mathcal{L}[\mathcal{D}^f u(t)] = f(\phi) \tilde{u}(\phi)$ and $\mathcal{L}[\Psi_t u(t)] = f(\phi)^{-1} \tilde{u}(\phi)$, so that $\mathcal{D}^f \Psi_t u(t) = \Psi_t \mathcal{D}^f u(t)$ for sufficiently smooth functions u . In view of (2.22), this shows that the operator (2.25) is the inverse of the operator (2.16) of Toaldo [51], when $b = 0$ and $M(t) = \bar{\nu}^*(t)$. Then heuristically (2.20) can be seen as the result of applying Φ_t to both sides of (2.13) with $A = \frac{1}{2} \frac{\partial^2}{\partial x^2}$. This will be made precise in Theorem 2.5. Finally, note that we also have $M(t) = u_{\sigma_f}(t)$ in view of (2.11).

Next we study the eigenstructure of the operator (2.16), and a corresponding property for (2.17), by considering continuous and exponentially bounded solutions $t \mapsto q(\lambda, t)$ for $\lambda \leq 0$ to the equations

$$\begin{cases} \frac{d}{dt} \int_0^t q(\lambda, s) \bar{\nu}(t-s) ds - \bar{\nu}(t) q(\lambda, 0) = \lambda q(\lambda, t), & t > 0, \\ q(\lambda, 0) = 1, \end{cases} \quad (2.26)$$

and

$$\begin{cases} \frac{d}{dt} q(\lambda, t) = \lambda \frac{d}{dt} \int_0^t q(\lambda, s) \bar{\nu}^*(t-s) ds, & t > 0, \\ q(\lambda, 0) = 1, \end{cases} \quad (2.27)$$

where $\bar{\nu}(s) = a + \nu(s, \infty)$ and $\bar{\nu}^*(s) = a^* + \nu^*(s, \infty)$. Note that in view of the discussion above the operator on the right-hand side of (2.27) coincides with the operator (2.21) studied in [31] if $\nu(0, \infty) = \infty$ and $M(t) = \bar{\nu}^*(t)$.

Theorem 2.1. *Let f be a special Bernstein function having representation (2.1) with $b = 0$, $\nu(0, \infty) = \infty$ with $s \rightarrow \bar{\nu}(s)$ absolutely continuous on $(0, \infty)$. Let f^* be the conjugate of f having representation (2.5). Let $L^f(t)$ be the inverse (2.12) of the Lévy subordinator $\sigma^f(t)$ with Laplace exponent f . Then for any $\lambda \leq 0$ the unique continuous and exponentially bounded solution to (2.26) is the moment generating function*

$$q(\lambda, t) = \mathbb{E}[e^{\lambda L^f(t)}] \quad (2.28)$$

and furthermore:

- (1) *This solution (2.28) to (2.26) is also the unique continuous and exponentially bounded solution to (2.27);*
- (2) *$[0, \infty) \ni \theta \mapsto q(-\theta, t)$ is completely monotone for each fixed $t \geq 0$, and $q(0, t) = 1$ for all $t \geq 0$;*
- (3) *$t \mapsto q(\lambda, t)$ is completely monotone, for each fixed $\lambda \leq 0$, if and only if $s \rightarrow \bar{\nu}(s)$ is completely monotone;*
- (4) *if $f(\phi)$ is regularly varying at $0+$ with some index $\rho \in [0, 1)$ then for all $\lambda < 0$,*

$$q(\lambda, t) \sim \frac{\bar{\nu}(t)}{a - \lambda} \quad \text{as } t \rightarrow \infty, \quad (2.29)$$

both $t \mapsto q(\lambda, t)$ and $\bar{\nu}(t)$ vary regularly at infinity with index $-\rho$, and

$$\int_0^\infty q(\lambda, t) dt = \infty \quad \text{for all } \lambda < 0. \quad (2.30)$$

Proof. First we prove that (2.28) solves (2.26). Since $\nu(0, \infty) = \infty$, [35, Thm 3.1] implies that $L^f(t)$ has a Lebesgue density $x \rightarrow l(x, t)$. Now [35, Eq. (3.13)] shows that

$$\begin{aligned} \mathcal{L}[l(x, \cdot)](\phi) &= \mathcal{L}\left[\frac{\partial}{\partial x} \Pr\{\sigma(x) \leq \cdot\}\right](\phi) \\ &= \frac{f(\phi)}{\phi} e^{-xf(\phi)} \end{aligned} \quad (2.31)$$

and therefore, for $\theta > 0$, we have [35, Corollary 3.5]

$$\tilde{l}(\theta, \phi) = \mathcal{L}[\mathcal{L}[l(x, t)](\phi)](\theta) = \frac{f(\phi)}{\phi} \frac{1}{\theta + f(\phi)}. \quad (2.32)$$

Using the convolution theorem for Laplace transform, and the fact that $b = 0$,

$$\begin{aligned} \mathcal{L} \left[\frac{d}{dt} \int_0^t q(\lambda, s) \bar{\nu}(t-s) ds - \bar{\nu}(t) \right] (\phi) &= \phi \mathcal{L} [q * \bar{\nu}] (\phi) - \frac{f(\phi)}{\phi} \\ &= f(\phi) \tilde{q}(\lambda, \phi) - \frac{f(\phi)}{\phi}. \end{aligned} \quad (2.33)$$

Taking Laplace transforms in (2.26) and solving for $\tilde{q}(\lambda, \phi)$ then yields

$$\mathcal{L} [q(\lambda, \cdot)] (\phi) = \tilde{q}(\lambda, \phi) = \frac{f(\phi)}{\phi} \frac{1}{f(\phi) - \lambda}. \quad (2.34)$$

Comparing (2.32) to (2.34) shows that the moment generating function of L^f is

$$\int_0^\infty e^{\lambda x} l(x, t) dx = q(\lambda, t) = \mathbb{E} e^{\lambda L^f(t)}, \quad \lambda \leq 0. \quad (2.35)$$

Since $t \mapsto q(\lambda, t)$ is the same function as [35, Eq. (3.17)], the proof of [35, Theorem 3.1] shows that $q(\lambda, t)$ is a continuous function of $t \geq 0$. Then it follows from the uniqueness of the Laplace transform (e.g., see Feller [18, Theorem 1, p. 430]) that (2.28) is the unique continuous and exponentially bounded solution to the problem (2.26), which proves the first part of the theorem.

Next we prove Item (1). Take Laplace transforms in (2.27) to get

$$\begin{aligned} \phi \tilde{q}(\lambda, \phi) - 1 &= \lambda \phi \mathcal{L} [q * \bar{\nu}^*] (\phi) \\ &= \lambda \phi (\phi^{-1} f^*(\phi) - b^*) \tilde{q}(\lambda, \phi). \end{aligned} \quad (2.36)$$

Since $f^*(\phi) = \phi/f(\phi)$, and $b^* = 0$ in view of (2.6), (2.36) can be rewritten

$$\tilde{u}(\lambda, \phi) = \frac{f(\phi)/\phi}{f(\phi) - \lambda} \quad (2.37)$$

which coincides with (2.34). This proves that (2.28) is also the unique continuous and exponentially bounded solution to (2.27), since they have the same Laplace transform.

Next we prove Item (2). Since $\nu(0, \infty) = \infty$, the subordinator $\sigma^f(t)$ is strictly increasing [46, Theorem 21.5], and hence $L^f(0) = 0$ a.s. Then we also have $q(\lambda, 0) = \mathbb{E} e^{\lambda L^f(0)} = 1$. Since $\theta \mapsto q(-\theta, t)$ is the Laplace transform of $x \mapsto l(x, t)$, it is completely monotone for each fixed $t \geq 0$.

Next we prove Item (3). If the function $s \rightarrow \bar{\nu}(s)$ is completely monotone, we have that for some measure $m(\cdot)$ and some non-negative constant a

$$\bar{\nu}(s) = a + \int_0^\infty e^{-sw} m(dw) = a + \int_s^\infty \int_0^\infty w e^{-yw} m(dw) dy \quad (2.38)$$

and therefore the function

$$y \rightarrow v(y) = \int_0^\infty e^{-yw} w m(dw) \quad (2.39)$$

is the completely monotone density of the Lévy measure $\nu(dy)$. This implies that f is a complete Bernstein function. Now [49, Thm 6.2 (vi)] implies that

$$\varphi(z) = \frac{z}{z - \lambda} \quad (2.40)$$

is a complete Bernstein function for $\lambda \leq 0$, and therefore $\varphi \circ f$ is a complete Bernstein function in view of [49, Corollary 7.9]. Therefore we have for some measure $k(\cdot)$ that

$$\varphi \circ f(\phi) = c + d\phi + \int_0^\infty (1 - e^{-\phi t}) \int_0^\infty e^{-ts} k(ds) dt \quad (2.41)$$

and therefore, integrating by parts in (2.41), one has

$$\begin{aligned} \frac{1}{\phi} (\varphi \circ f) &= \int_0^\infty e^{-\phi t} \left(c + d\phi + \int_t^\infty \int_0^\infty e^{-ws} k(ds) dw \right) \\ &= \int_0^\infty e^{-\phi t} \left(c + d\phi + \int_0^\infty s^{-1} e^{-st} k(ds) \right) dt. \end{aligned} \quad (2.42)$$

Since by (2.34) we also have

$$\frac{1}{\phi} (\varphi \circ f) = \int_0^\infty e^{-\phi t} q(\lambda, t) dt, \quad (2.43)$$

and since $t \mapsto c + d\phi + \int_t^\infty \int_0^\infty e^{-ws} k(ds) dw$ is obviously continuous, it follows from the uniqueness theorem for Laplace transforms that

$$q(\lambda, t) = c + d\phi + \int_0^\infty s^{-1} e^{-st} k(ds) \quad (2.44)$$

and if $d = 0$ this proves that $t \rightarrow q(\lambda, t)$ is completely monotone, which establishes the direct half of Item (3). The fact that $d = 0$ can be ascertained by observing that

$$\frac{1}{\phi} (\varphi \circ f) = \frac{1}{\phi} \frac{f(\phi)}{f(\phi) - \lambda} \rightarrow d$$

as $\phi \rightarrow \infty$ by [49, p. 23, Item (iv)] and that $f \geq 0$, $f' \geq 0$, and $-\lambda \geq 0$. Then $f(\phi) \rightarrow L \in (0, \infty]$ as $\phi \rightarrow \infty$. Hence

$$\frac{f(\phi)}{f(\phi) - \lambda} \rightarrow \frac{L}{L - \lambda} \leq 1 \quad \text{as } \phi \rightarrow \infty.$$

It follows that $d = 0$. In view of (2.34) we have

$$\frac{1}{\phi} \frac{f(\phi)}{f(\phi) - \lambda} = \int_0^\infty e^{-\phi t} q(\lambda, t) dt \quad (2.45)$$

and hence

$$\begin{aligned} G(\phi) &:= \frac{1}{\phi} \frac{f(\phi)}{f(\phi) - \lambda} = \int_0^\infty e^{-\phi t} \left(c + \int_0^\infty s^{-1} e^{-ts} k(ds) \right) dt \\ &= \frac{c}{\phi} + \int_0^\infty \frac{1}{\phi + s} s^{-1} k(ds) \end{aligned} \quad (2.46)$$

which is a Stieltjes function provided that $\int_0^\infty (1 + s)^{-1} s^{-1} k(ds) < \infty$. But such an integral converges since by (2.44) and the fact that $q(\lambda, 0) = 1$, it must be true that

$$1 = q(\lambda, 0) = c + \int_0^\infty s^{-1} k(ds) \quad (2.47)$$

and therefore $s^{-1}k(ds)$ is integrable. Now note that $F(\phi) = 1/G(\phi)$ is a complete Bernstein function by [49, Thm 7.3]. Then $\phi/F(\phi) = \phi G(\phi)$ is also complete by [49, Proposition 7.1]. It follows that

$$\frac{1}{\phi G(\phi)} = \frac{f(\phi) - \lambda}{f(\phi)} = 1 - \frac{\lambda}{f(\phi)}$$

is a Stieltjes function in view of [49, Thm 7.3]. Let $g(\phi) := 1 - \lambda/f(\phi)$ and use (2.7) to write

$$g(\phi) = \frac{\mathfrak{a}}{\phi} + \mathfrak{b} + \int_0^\infty \frac{1}{\phi + s} \mathfrak{k}(ds) \quad (2.48)$$

then

$$-\lambda/f(\phi) = g(\phi) - 1 = \frac{\mathfrak{a}}{\phi} + \mathfrak{b} - 1 + \int_0^\infty \frac{1}{\phi + s} \mathfrak{k}(ds) \quad (2.49)$$

and since we know that $-\lambda/f(\phi)$ is non-negative (recall that $f(\phi) \geq 0$ and $\lambda \leq 0$) also (2.49) must be non-negative for all $\phi \in (0, \infty)$. In particular by letting $\phi \rightarrow \infty$ we deduce that $\mathfrak{b} \geq 1$. We have thus proved that $-\lambda/f(\phi)$ is a Stieltjes function. Therefore by applying again [49, Thm 7.3] to the Stieltjes function $-\lambda/f$ we deduce that $f(\phi)$ is a complete Bernstein function and therefore

$$f(\phi) = a + \int_0^\infty (1 - e^{-\phi s}) \nu(s) ds \text{ with } \nu(s) = \int_0^\infty e^{-st} \mathfrak{m}(dt) \quad (2.50)$$

for some measure \mathfrak{m} and some $a \geq 0$ (since we are assuming $b = 0$). From (2.50) we get that

$$\bar{\nu}(s) = a + \int_s^\infty \nu(w) dw = a + \int_0^\infty e^{-st} \frac{\mathfrak{m}(dt)}{t} \quad (2.51)$$

and this proves that $s \rightarrow \bar{\nu}(s)$ is completely monotone, which establishes the converse part of Item (3).

Finally we prove Item (4). We say that a Borel measurable function $f : (0, \infty) \rightarrow [0, \infty)$ varies regularly at infinity with index $\rho \in \mathbb{R}$ if

$$\lim_{x \rightarrow \infty} \frac{f(cx)}{f(x)} = c^\rho \quad (2.52)$$

for any $c > 0$, see for example Bingham et al. [12, p. 1]. It follows that, for any $\varepsilon > 0$, for some $x_0 > 0$, we have [18, Lemma VIII.8.2]

$$x^{\rho-\varepsilon} < f(x) < x^{\rho+\varepsilon} \text{ for all } x \geq x_0. \quad (2.53)$$

If $\rho = 0$, we say that f is slowly varying. If $f(1/x)$ is regularly varying at infinity with index $-\rho$, then we say that f is regularly varying at zero with index ρ . Suppose that $U(x)$ is a nondecreasing right-continuous function on $[0, \infty)$ with Laplace transform

$$\tilde{U}(s) = \int_0^\infty e^{-sx} U(dx)$$

for all $s > 0$. The Karamata Tauberian Theorem [18, Thm XIII.5.1] states that

$$U(x) \sim \frac{x^\rho L(x)}{\Gamma(1+\rho)} \text{ as } x \rightarrow \infty \iff \tilde{U}(s) \sim s^{-\rho} L(1/s) \text{ as } s \rightarrow 0, \quad (2.54)$$

where $L(x)$ is slowly varying at infinity and $\rho \geq 0$.

Suppose that $f(\phi)$ varies regularly at $x = 0$ with index $\rho = 1 - \beta$ for some $\beta \in (0, 1]$. Note that if f varies regularly at zero, we must have $\rho = 1 - \beta$ for some $\beta \in [0, 1]$ due to the Lévy-Khintchine representation (2.1) [10, Proposition 1.5]. If $\rho > 0$, it follows from (2.53) that we must have $f(0+) = 0$, and hence $a = 0$ in (2.1). If $\rho = 0$, then $a > 0$ is possible, in which case $f(0+) = a$. In either case, from (2.34) we have

$$\tilde{q}(\lambda, \phi) = \frac{f(\phi)}{\phi} \frac{1}{f(\phi) - \lambda} \sim \frac{f(\phi)}{\phi} \frac{1}{a - \lambda} \quad \text{as } \phi \rightarrow 0+$$

where $\lambda < 0$, and then it is easy to check that $\phi \mapsto \tilde{q}(\lambda, \phi)$ varies regularly at $\phi = 0+$ with index $-\beta$. Define

$$Q(\lambda, t) = \int_0^t q(\lambda, s) ds$$

so that

$$\tilde{q}(\lambda, \phi) = \int_0^\infty e^{-\phi t} Q(\lambda, dt).$$

Apply the Karamata Tauberian Theorem to see that $t \mapsto Q(\lambda, t)$ varies regularly at infinity with index β , and furthermore that

$$Q(\lambda, t) \sim \frac{tf(1/t)}{\Gamma(1 + \beta)(a - \lambda)} \quad \text{as } t \rightarrow \infty.$$

Now apply the Monotone Density Theorem [12, Thm 1.7.2] to see that $t \mapsto q(\lambda, t)$ varies regularly at infinity with index $\beta - 1$, and furthermore $tq(\lambda, t)/Q(\lambda, t) \rightarrow \beta$ as $t \rightarrow \infty$, so that

$$q(\lambda, t) \sim \frac{\beta f(1/t)}{\Gamma(1 + \beta)(a - \lambda)} \quad \text{as } t \rightarrow \infty. \quad (2.55)$$

Next observe that (2.3), along with the fact that $b = 0$, implies that $f(\phi)/\phi$ is the Laplace transform of $\bar{\nu}(t)$. Then another application of the Karamata Tauberian Theorem shows that

$$\bar{\nu}(t) \sim \frac{\beta f(1/t)}{\Gamma(1 + \beta)} \quad \text{as } t \rightarrow \infty. \quad (2.56)$$

Combining (2.55) and (2.56) shows that $(a - \lambda)q(\lambda, t) \sim \bar{\nu}(t)$ as $t \rightarrow \infty$, which proves the first statement of Item (4).

Finally, since $t \mapsto Q(\lambda, t)$ varies regularly at infinity with index $\beta > 0$, it follows from (2.53) that $Q(\lambda, t) \rightarrow \infty$ as $t \rightarrow \infty$, which proves the second statement of Item (4). \square

Remark 2.2. If $s \rightarrow \bar{\nu}(s)$ is completely monotone, then we showed in the proof above that

$$\bar{\nu}(s) = a + \int_0^\infty e^{-sw} m(dw) = a + \int_s^\infty \int_0^\infty w^{-1} e^{-yw} m(dw) dy \quad (2.57)$$

and therefore the Lévy density of ν is also completely monotone. The corresponding Bernstein function f is thus a complete Bernstein function. Therefore the adjoint f^* is also complete (Proposition 7.1 in [49]) and has a Lévy density which is completely monotone with tail

$$\bar{\nu}^*(s) = a^* + \int_s^\infty \int_0^\infty e^{-tw} m(dw) dt$$

$$= a^* + \int_0^\infty w^{-1} e^{-sw} m(dw) \quad (2.58)$$

for some measure m . Therefore $s \rightarrow \bar{\nu}^*(s)$ is a completely monotone function and Item (3) of Theorem 2.1 may be restated as: the function $t \rightarrow q(\lambda, t)$ is completely monotone if and only if $s \rightarrow \bar{\nu}^*(s)$ is completely monotone. That is, $\bar{\nu}$ is completely monotone if and only if $\bar{\nu}^*$ is completely monotone.

Remark 2.3. It follows from [35, Theorem 3.1] that the inverse subordinator (2.12) has a probability density

$$l(x, t) = \int_0^t \bar{\nu}(t-s) \mu(ds, x)$$

for any $t > 0$, where $\mu(ds, x)$ is the probability distribution of the subordinator $\sigma^f(x)$ with Laplace symbol (2.1), and $\bar{\nu}(s) = a + \nu(s, \infty)$. It follows that we can also write

$$q(\lambda, t) = \int_0^t \int_0^\infty e^{\lambda x} \bar{\nu}(t-s) \mu(ds, x) dx \quad (2.59)$$

in view of (2.28).

Generalized relaxation equations and patterns have been also examined in [24; 25; 26; 53; 54]. Kochubei [26] considered operators similar to that appearing in (2.26) but with different kernels of convolution. By making assumptions on the Laplace transform of such kernels he determined sufficient conditions for the complete monotonicity of the solution. In [24; 25] he also studied distributed-order relaxation patterns, i.e., the solution to

$$\int_0^1 \frac{\partial^\alpha}{\partial t^\alpha} u \mu(\alpha) d\alpha = \lambda u, \quad \lambda < 0, \quad (2.60)$$

where μ is a non-negative continuous function on $[0, 1]$. He pointed out that in this case the relaxation pattern is completely monotone. Observe that (2.60) is a particular case of (2.27) (see [52] for details on this point). An important application of (2.60) is to ultraslow relaxation where $f(\phi)$ is slowly varying at $\phi = 0$, see [34] for more details.

Remark 2.4. The proof of [34, Theorem 3.9] provides a partial converse of Item (4) in Theorem 2.1 in the case of ultraslow diffusion. If the tail of the Lévy measure is of the form

$$\bar{\nu}(t) = \int_0^1 t^{-\eta} p(\eta) d\eta$$

where p varies regularly at zero with some index $\alpha > -1$, then $\bar{\nu}(t)$ is slowly varying [34, Lemma 3.1], and then it follows that the Laplace symbol

$$f(\phi) = \int_0^1 \Gamma(1-\eta) t^{-\eta} p(\eta) d\eta$$

is also slowly varying [34, Eq. (3.18)].

2.3. Time-changed processes. Theorem 2.1 and the discussion above suggests how the approaches of Meerschaert and Scheffler [35], Toaldo [51], and Magdziarz and Schilling [31] may be rearranged under a unifying framework, by resorting to special Bernstein functions. Now we extend the equations (2.26) and (2.27) to a more general form. Suppose that A is a self-adjoint, dissipative operator that generates a C_0 -semigroup of operators T_t on the Hilbert space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$, and consider the generalized abstract Cauchy problem

$$\frac{d}{dt} \int_0^t g(s) \bar{v}(t-s) ds - \bar{v}(t)g(0) = Ag(t) \quad (2.61)$$

or equivalently (as we will show in Theorem 2.5)

$$\frac{d}{dt}g(t) = \frac{d}{dt} \int_0^t Ag(s) \bar{v}^*(t-s) ds. \quad (2.62)$$

If $A = \frac{1}{2} \frac{\partial^2}{\partial x^2}$ then (2.62) with $g(t) = q(x, t)$ reduces to the equation (2.20) in Magdziarz and Schilling [31].

We follow Kolokoltsov [28, Section 1.9] and Schilling et al. [49, Chapter 12] for the basic theory of semigroups and generators. See Jacob [22, Chapter 2] or [49, Chapter 11] for a nice summary of the classical theory of linear self-adjoint operators on Hilbert spaces. By the definition of a C_0 -semigroup we have for all $u \in \mathfrak{H}$ that

- (1) $T_0 u = u$
- (2) $T_t T_s u = T_{t+s} u$
- (3) $\lim_{t \rightarrow 0} \|T_t u - u\|_{\mathfrak{H}} = 0$.

Note that since A is a self-adjoint generator and it is dissipative we have that the spectrum is non-positive, i.e. for any $u \in \text{Dom}(A)$ we have $\langle Au, u \rangle \leq 0$ [49, Proposition 11.2 and formula (11.4)], and we can apply the spectral theorem [49, Thm 11.4]. Therefore we know that there exists an orthogonal projection-valued measure

$$E(B) := \int_B E(d\lambda) \quad (2.63)$$

for Borel sets $B \subseteq \mathbb{R}$, supported on the spectrum of A and therefore in this case on a subset of $(-\infty, 0]$, such that given a function

$$\Xi : (-\infty, 0] \rightarrow \mathbb{R} \quad (2.64)$$

we may write [49, Eq. (11.10)]

$$\Xi(A)u = \int_{(-\infty, 0]} \Xi(\lambda) E(d\lambda)u \quad (2.65)$$

for $u \in \text{Dom}(\Xi(A))$, where by [49, Eq. (11.11)] we have

$$\text{Dom}(\Xi(A)) = \left\{ u \in \mathfrak{H} : \int_{(-\infty, 0]} |\Xi(\lambda)|^2 \langle E(d\lambda)u, u \rangle < \infty \right\}. \quad (2.66)$$

Therefore given any $u \in \mathfrak{H}$ we can write

$$T_t u = e^{At} u = \int_{(-\infty, 0]} e^{\lambda t} E(d\lambda)u. \quad (2.67)$$

Then for all $u \in \mathfrak{H}$ we have

$$\|T_t u\|_{\mathfrak{H}} = \left\| \int_{(-\infty, 0]} e^{\lambda t} E(d\lambda) u \right\|_{\mathfrak{H}} \leq \left\| \int_{(-\infty, 0]} E(d\lambda) u \right\|_{\mathfrak{H}} = \|u\|_{\mathfrak{H}}, \quad (2.68)$$

so that T_t is a contraction semigroup (e.g., see [49, Example 11.5]). Since T_t is a C_0 -semigroup we also know that for $u \in \text{Dom}(A)$ we have [28, Thm 1.9.1]

$$\frac{d}{dt} T_t u = A T_t u = T_t A u \quad (2.69)$$

and that the map $t \rightarrow T_t u$ is the unique classical solution to the abstract Cauchy problem [17, Proposition 6.2]

$$\begin{cases} \frac{d}{dt} g(t) = A g(t), \\ g(0) = u. \end{cases} \quad (2.70)$$

Theorem 2.5. *Let f and $q(\lambda, t)$ be as in Theorem 2.1. Let $x \rightarrow l(x, t)$ be the density of inverse process (2.12) of the subordinator $\sigma^f(t)$ with Laplace symbol f . Let T_t be a C_0 -semigroup on the Hilbert space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ whose generator A is self-adjoint and dissipative. The unique solution to (2.61) on \mathfrak{H} subject to $g(0) = u \in \text{Dom}(A)$ coincides with the solution of (2.62). This solution is the function $q(A, t)u$ defined in the sense of (2.65) for all $u \in \mathfrak{H}$, and we also have*

$$q(A, t)u = \int_0^\infty T_s u l(s, t) ds, \quad (2.71)$$

a Bochner integral on \mathfrak{H} .

Proof. Using the “functional calculus” approach introduced above we define

$$q(A, t)u = \int_{(-\infty, 0]} q(\lambda, t) E(d\lambda) u. \quad (2.72)$$

Now we recall from Theorem 2.1 that the function $[0, \infty) \ni \theta \rightarrow q(-\theta, t)$ is completely monotone and may be written as the Laplace transform of the density $x \rightarrow l(x, t)$ of the inverse process (2.12) of the subordinator $\sigma^f(t)$ with Laplace symbol f .

Therefore (2.72) becomes

$$\begin{aligned} q(A, t)u &= \int_{(-\infty, 0]} q(\lambda, t) E(d\lambda) u \\ &= \int_{(-\infty, 0]} \int_0^\infty e^{\lambda s} l(s, t) ds E(d\lambda) u \\ &= \int_0^\infty \int_{(-\infty, 0]} e^{\lambda s} E(d\lambda) u l(s, t) ds \\ &= \int_0^\infty T_s u l(s, t) ds \end{aligned} \quad (2.73)$$

by (2.67). Note that (2.73) holds for any function $u \in \mathfrak{H}$: indeed [49, Example 11.5]

$$\|q(A, t)u\|_{\mathfrak{H}} = \left\| \int_0^\infty T_s u l(s, t) ds \right\|_{\mathfrak{H}} \leq \int_0^\infty \|T_s u\|_{\mathfrak{H}} l(s, t) ds \leq \|u\|_{\mathfrak{H}} \quad (2.74)$$

using (2.68) along with $\int_0^\infty l(s, t) ds = 1$. The fact that $q(A, t)$ maps $\text{Dom}(A)$ into itself may be ascertained by observing that for $u \in \text{Dom}(A)$

$$\begin{aligned} Aq(A, t)u &= \int_{(-\infty, 0]} \lambda E(d\lambda) \int_{(-\infty, 0]} q(\mu, t) E(d\mu) u \\ &= \int_{(-\infty, 0]} \lambda q(\lambda, t) E(d\lambda) u \\ &= \int_{(-\infty, 0]} q(\mu, t) E(d\mu) \int_{(-\infty, 0]} \lambda E(d\lambda) u = q(A, t) Au \end{aligned} \quad (2.75)$$

where we used the fact that for all Borel sets B, B' it is true that $E(B)E(B') = E(B \cap B')$ and that $E(B)$ maps $\text{Dom}(A)$ into itself [49, Thm 11.4]. The fact that $q(A, t)$ solves (2.61) and (2.62) can be ascertained as follows. By using again (2.65) and Theorem 2.1 we have that

$$\begin{aligned} &\frac{d}{dt} \int_0^t q(A, s) \bar{\nu}(t-s) ds - \bar{\nu}(t) q(A, 0) \\ &= \int_{(-\infty, 0]} \frac{d}{dt} \int_0^t q(\lambda, s) \bar{\nu}(t-s) ds E(d\lambda) - \int_{(-\infty, 0]} \bar{\nu}(t) q(\lambda, 0) E(d\lambda) \\ &= \int_{(-\infty, 0]} \left(\frac{d}{dt} \int_0^t q(\lambda, s) \bar{\nu}(t-s) ds - \bar{\nu}(t) q(\lambda, 0) \right) E(d\lambda) \\ &= \int_{(-\infty, 0]} \lambda q(\lambda, t) E(d\lambda) \\ &= Aq(A, t) \end{aligned} \quad (2.76)$$

as well as

$$\begin{aligned} \frac{d}{dt} q(A, t) &= \int_{(-\infty, 0]} \frac{d}{dt} q(\lambda, t) E(d\lambda) u \\ &= \int_{(-\infty, 0]} \frac{d}{dt} \int_0^t \lambda q(\lambda, s) \bar{\nu}^*(t-s) ds E(d\lambda) u \\ &= \frac{d}{dt} \int_0^t Aq(A, s) \bar{\nu}^*(t-s) ds. \end{aligned} \quad (2.77)$$

Finally we prove uniqueness. From [51, Eq. (5.13)] it follows that, for any $u \in \text{Dom}(A)$, the solution $q(t)$ of the generalized Cauchy problem

$$\mathfrak{D}_t^f q = Aq; \quad q(0) = u \quad (2.78)$$

has Laplace transform ($t \mapsto \lambda$)

$$\tilde{q}(\lambda) = (f(\lambda) - A)^{-1} \frac{f(\lambda)}{\lambda} u. \quad (2.79)$$

In view of (2.17) the generalized Cauchy problem (2.78) is another way to write (2.61), and hence (2.79) also holds for any solution to (2.61), for any $u \in \text{Dom}(A)$. The remainder of the argument is due to Baeumer [6]. Since A generates a C_0 -semigroup, the resolvent $(f(\lambda) - A)^{-1}$ is a bounded operator for all $f(\lambda)$ in the right half plane. In particular $(f(\lambda) - A)^{-1}0 = 0$ and hence by the uniqueness of the Laplace transform, we have $q = 0$ for initial data $u = 0$. Then, given two solutions q_1, q_2 to (2.78), their difference $q = q_1 - q_2$ solves (2.78) with $u = 0$, and hence $q_1 = q_2$. Therefore, the solution to (2.61) is unique. An argument similar to

(2.37) shows that the solution to (2.62) for any $u \in \text{Dom}(A)$ has the same Laplace transform (2.79), hence it is also unique. \square

Remark 2.6. Fractional Cauchy problems of the form (2.15) with $p(x, 0) = f(x) \in \text{Dom}(A)$ were considered by Bazhlekova [7] and Baeumer and Meerschaert [3]. In this case, we have $f(\phi) = \phi^\beta$ for some $0 < \beta < 1$. Distributed order fractional Cauchy problems with

$$f(\phi) = \int_0^1 \phi^\beta p(\beta) d\beta$$

were considered by Mijena and Nane [43] and Bazhlekova [8]. Solutions to the generalized Cauchy problem (2.13), which is equivalent to (2.61) or (2.62), were developed by Toaldo [51].

3. SEMI-MARKOV DYNAMICS

In this section we construct a semi-Markov process (3.6) on a countable state space whose dynamics are governed by the operator equations

$$\frac{d}{dt} \int_0^t g(s) \bar{\nu}(t-s) ds - \bar{\nu}(t)g(0) = Ag(t) \quad (3.1)$$

and

$$\frac{d}{dt}g(t) = \frac{d}{dt} \int_0^t Ag(s) \bar{\nu}^*(t-s) ds \quad (3.2)$$

where A is an $|\mathcal{S}| \times |\mathcal{S}|$ matrix (we allow a countably infinite state space $|\mathcal{S}| = \infty$) and $g(t) = g(A, t)$ is the operator of Thm 2.5 defined by (2.72) using functional calculus. We will work all throughout this section under the following assumptions.

- A1) $X(t)$ is a continuous-time Markov chain with countable state-space \mathcal{S} , generated by A and associated to the semigroup of matrices $\{P_t\}_{t \geq 0}$. We assume that A is symmetric, and that for its elements $a_{i,j}$ it is true that $\sup \{-a_{i,i}\} < \infty$. The assumption $\sup \{-a_{i,i}\} < \infty$ implies that $X(t)$ is non explosive [44, Thm 2.7.1]. Furthermore within such a framework it is true that P_t solves the so-called Kolmogorov backward equation [44, Thm 2.8.3]

$$\frac{d}{dt}P_t = AP_t, \quad (3.3)$$

and also the forward one [44, Thm 2.8.6]

$$\frac{d}{dt}P_t = P_t A, \quad (3.4)$$

both subject to $P_0 = \mathbf{1}$. If \mathcal{S} is finite then A is a finite matrix (hence A is bounded) and the representation $P_t = e^{At}$ is true. Since we do not assume that \mathcal{S} is finite (but only countable) we can use the fact that A is symmetric and therefore the representation $P_t = e^{At}$ is true in the sense of (2.65) which becomes in this case

$$P_t = e^{At} = \sum_j e^{\lambda_j t} v_j v_j' \quad (3.5)$$

where λ_j are the eigenvalues of A and the v_j are a orthonormal basis of eigenvectors of A .

- A2) Y_n is a (homogeneous) discrete-time Markov chain on the countable state space \mathcal{S} with symmetric transition matrix H and we denote by $h_{i,j}$ the elements of H .
- A3) J_i are i.i.d. r.v.'s with c.d.f. $F_J(t) = 1 - q(\lambda, t)$ where $t \rightarrow q(\lambda, t)$ is completely monotone by Thm 2.1. Assume also that the conditions of Item 4 are fulfilled. We define $T_n = \sum_{i=1}^n J_i$ and

$$Y(t) = Y_n, \text{ for } T_n \leq t < T_{n+1} \quad (3.6)$$

and assume that the i.i.d. r.v.'s J_i are also independent from Y_n and therefore T_n and Y_n are independent. Observe that $\lim_{t \rightarrow \infty} F_J(t) = 1 - a$ and if $a > 0$ $F_J(t)$ is not a possibly defective c.d.f.. For the sake of simplicity we work with $a = 0$. We also assume that λ is strictly negative because if $\lambda = 0$ the r.v.'s J_i become degenerate constant r.v.'s assuming value zero.

- A4) With $\sigma^f(t)$ we denote the subordinator with Laplace exponent (2.1) for $b = 0$. Since in A3) we assumed that $t \rightarrow q(\lambda, t)$ is that of Thm 2.1 (and is completely monotone) we must have that $\nu(0, \infty) = \infty$ and that $s \rightarrow \bar{\nu}(s)$ is completely monotone. Given a subordinator $\sigma^f(t)$, $t \geq 0$, with Laplace symbol f we denote the inverse process (2.12) by $L^f(t)$, $t \geq 0$. Furthermore we assume that $f(\phi)$ is regularly varying at $0+$ for some index $\rho \in [0, 1)$ and therefore by Item 4 of Theorem 2.1 the functions $t \rightarrow \bar{\nu}(t)$ and $t \rightarrow q(\lambda, t)$ are regularly varying at infinity with index $-\rho$, and $\mathbb{E}J_i = \infty$ for all i .

Note that equations (3.1) and (3.2) generalize the Kolmogorov backward equation (3.3) to semi-Markov processes. The corresponding generalizations of (3.4) are

$$\frac{d}{dt} \int_0^t g(s) \bar{\nu}(t-s) ds - \bar{\nu}(t)g(0) = g(t)A \quad (3.7)$$

and

$$\frac{d}{dt}g(t) = \frac{d}{dt} \int_0^t g(s)A \bar{\nu}^*(t-s)ds. \quad (3.8)$$

Corollary 3.1. *Let A be as in A1). The matrix $q(A, t)$, defined in the sense of (2.65), where $q(\lambda, t)$ is the function of Theorem 2.1 is the unique solution to (3.1) and (3.7) as well as (3.2) and (3.8) with initial datum $g(0) = \mathbf{1}$ (identity matrix). Furthermore we have, using the Bochner integral, that*

$$q(A, t) = \int_0^\infty P_s l(s, t) ds. \quad (3.9)$$

Proof. This Corollary is a direct consequence of Thm 2.5. To clarify the arguments, here we provide some details where the proof becomes simpler in the present case. Observe that now the generator A is a matrix with non-positive eigenvalues and, for an orthonormal basis v_j of eigenvectors of A , the spectral representation (2.65) here is the matrix

$$\Xi(A) = \sum_j \Xi(\lambda_j) v_j v_j'. \quad (3.10)$$

The function $q(A, t)$ is therefore the matrix

$$q(A, t) = \sum_j q(\lambda_j, t) v_j v_j'$$

$$\begin{aligned}
&= \sum_j \int_0^\infty e^{\lambda_j s} l(s, t) ds v_j v'_j \\
&= \int_0^\infty P_s l(s, t) ds.
\end{aligned} \tag{3.11}$$

It is clear that equation (3.7) is also satisfied since $P_t A = A P_t$. In our case this may be easily checked

$$\begin{aligned}
Aq(A, t) &= \sum_j \lambda_j v_j v'_j \sum_i \int_0^\infty e^{\lambda_i s} l(s, t) ds v_i v'_i \\
&= \int_0^\infty \sum_j \sum_i \lambda_j e^{\lambda_i s} v_j v'_j v'_i l(s, t) ds \\
&= \sum_j \int_0^\infty \lambda_j e^{\lambda_j s} v_j v'_j l(s, t) ds \\
&= \sum_j \int_0^\infty e^{\lambda_j s} v_j v'_j l(s, t) ds \sum_i \lambda_i v_i v'_i \\
&= q(A, t) A
\end{aligned} \tag{3.12}$$

which finishes the proof. \square

Theorem 3.2. *The process $Y(t)$ introduced in A3) is semi-Markov and such that the $|\mathcal{S}| \times |\mathcal{S}|$ matrix with elements*

$$q_{i,j}(t) = P\{Y(t) = j | Y(0) = i\} \tag{3.13}$$

satisfies (3.1) and (3.7) as well as (3.2) and (3.8) with initial datum $g(0) = \mathbf{1}$ for $A = -\lambda(H - \mathbf{1})$, where H is the transition matrix of the discrete-time Markov chain Y_n on \mathcal{S} introduced in A2). Furthermore

$$(q_{i,j}(t))_{i,j} = q(A, t) = \int_0^\infty e^{-\lambda(H - \mathbf{1})s} l(s, t) ds \tag{3.14}$$

where $q(A, t)$ is the function of Thm 2.5 and $s \rightarrow l(s, t)$ is the density of the process $L^f(t)$ in A4).

Proof. First note that since $Y(t) = Y_n$, for $T_n \leq t < T_{n+1}$, we have that $Y(t) = Y_{N^\vee(t)}$ where

$$N^\vee(t) = \max\{n \in \mathbb{N} : T_n \leq t\} \tag{3.15}$$

and therefore $Y(t)$ is a semi-Markov process since

$$P(Y_n = j, J_n \leq t | (Y_0, T_0) \cdots (Y_{n-1}, T_{n-1})) = P(Y_n = j | Y_{n-1} = i) (1 - q(\lambda, t)) \tag{3.16}$$

due to the independence between the r.v.'s J_i and the chain Y_n . Therefore the $q_{i,j}$ satisfy the (backward) renewal equation [15, Chapter 10, formula (5.5)]

$$q_{i,j}(t) = q(\lambda, t) \delta_{i,j} + \sum_{l \in \mathcal{S}} h_{i,l} \int_0^t q_{l,j}(s) f_J(t-s) ds \tag{3.17}$$

where $h_{i,j}$ are the elements of the symmetric transition matrix H of the discrete-time chain.

Next we prove that $t \rightarrow q_{i,j}(t)$ is continuous. Since $t \rightarrow q(\lambda, t)$ is completely monotone under A3), we can write

$$q(\lambda, t) = \int_0^\infty e^{-tw} \frac{m(dw)}{w} \quad (3.18)$$

for some measure $m(dw)$, and hence we also have

$$\mathfrak{f}_J(t) = \frac{d}{dt} (1 - q(\lambda, t)) = \int_0^\infty e^{-tw} m(dw).$$

Then (3.17) yields

$$q_{i,j}(t) = q(\lambda, t) \delta_{i,j} + \int_0^t \sum_{l \in \mathcal{S}} h_{i,l} q_{l,j}(s) \int_0^\infty e^{-(t-s)w} m(dw) ds. \quad (3.19)$$

Since $|\sum_{l \in \mathcal{S}} h_{i,l} q_{l,j}(w)| \leq \sum_{l \in \mathcal{S}} h_{i,l} = 1$, and since $\mathfrak{f}_J(0)$ is a positive and decreasing function, it follows that for any $i, j \in \mathcal{S}$ and $h > 0$ we have

$$\begin{aligned} & |q_{i,j}(t) - q_{i,j}(t+h)| \\ & \leq |q(\lambda, t) - q(\lambda, t+h)| + \left| \int_0^t \sum_{l \in \mathcal{S}} h_{i,l} q_{l,j}(w) \int_0^\infty e^{-(t-s)w} (1 - e^{-th}) m(dw) ds \right| \\ & \quad + \left| \int_t^{t+h} \sum_{l \in \mathcal{S}} h_{i,l} q_{l,j}(w) \int_0^\infty e^{-(t+h-s)w} m(dw) ds \right| \\ & \leq q(\lambda, t) - q(\lambda, t+h) + \int_0^t (q(\lambda, t-s) - q(\lambda, t+h-s)) ds + \mathfrak{f}_J(0)h. \end{aligned} \quad (3.20)$$

Then the first and the last terms in (3.20) go to zero as $h \rightarrow 0+$ while for the integral we note that

$$(q(\lambda, t-s) - q(\lambda, t+h-s)) \leq q(\lambda, t-s) \quad (3.21)$$

and $\int_0^t q(\lambda, t-s) ds < \infty$ for all $t < \infty$. Therefore the integral in (3.20) goes to zero as $h \rightarrow 0+$ by dominated convergence theorem. For $h < 0$ the argument is similar. Hence $t \rightarrow q_{i,j}(t)$ is continuous.

Therefore the $q_{i,j}(t)$, $i, j \in \mathcal{S}$, are the unique continuous functions whose Laplace transforms satisfy (we here use (2.33) and (3.17))

$$\tilde{q}_{i,j}(\phi) = \frac{f(\phi)/\phi}{f(\phi) - \lambda} \delta_{i,j} - \sum_{l \in \mathcal{S}} h_{i,l} \tilde{q}_{l,j}(\phi) \frac{\lambda}{f(\phi) - \lambda}. \quad (3.22)$$

Now multiply by $f(\phi) - \lambda$ on both sides of (3.22) and subtract $q_{i,j}(0) = \delta_{i,j}$ to get

$$f(\phi) \tilde{q}_{i,j}(\phi) - \frac{f(\phi)}{\phi} q_{i,j}(0) = -\lambda \left(\sum_{l \in \mathcal{S}} \tilde{q}_{l,j}(\phi) h_{i,l} - \tilde{q}_{i,j}(\phi) \right). \quad (3.23)$$

Now multiply by $f^*(\phi) = \phi/f(\phi)$ to get

$$\phi \tilde{q}_{i,j}(\phi) - q_{i,j}(0) = -\lambda f^*(\phi) \left(\sum_{l \in \mathcal{S}} \tilde{q}_{l,j}(\phi) h_{i,l} - \tilde{q}_{i,j}(\phi) \right). \quad (3.24)$$

By Laplace inversion of (3.23) and (3.24) we have using (2.3) that $q_{i,j}(t)$ satisfies for all $i, j \in \mathcal{S}$

$$\begin{cases} \frac{d}{dt} \int_0^t q_{i,j}(s) \bar{\nu}(t-s) ds - \bar{\nu}(t) q_{i,j}(0) = -\lambda (\sum_{l \in \mathcal{S}} q_{l,j}(t) h_{i,l} - q_{i,j}(t)) \\ q_{i,j}(0) = \delta_{i,j} \end{cases} \quad (3.25)$$

and

$$\begin{cases} \frac{d}{dt} q_{i,j}(t) = -\lambda \frac{d}{dt} \int_0^t (\sum_{l \in \mathcal{S}} q_{l,j}(s) h_{i,l} - q_{i,j}(s)) \bar{\nu}^*(t-s) ds \\ q_{i,j}(0) = \delta_{i,j}. \end{cases} \quad (3.26)$$

Note that the solution to the matrix problem, for $A = -\lambda(H - \mathbf{1})$,

$$\begin{cases} \frac{d}{dt} \int_0^t q(A, s) \bar{\nu}(t-s) ds - \bar{\nu}(t) q(A, 0) = A q(A, t) \\ q(A, 0) = \mathbf{1} \end{cases} \quad (3.27)$$

is a matrix such that each entry satisfy the backward equation (3.25). The backward equation therefore is proved.

The forward equation follows by (2.75) or (3.12). To verify in this special case, since H is symmetric then so is $H - \mathbf{1}$ and the eigenvalues of $H - \mathbf{1}$ are non-positive since H is a transition matrix. Therefore we can write as in (3.11)

$$q_{-\lambda(H-\mathbf{1})}(t) = \int_0^\infty e^{-\lambda(H-\mathbf{1})s} l(s, t) ds \quad (3.28)$$

and we know that

$$-\lambda(H - \mathbf{1}) q_{-\lambda(H-\mathbf{1})}(t) = q_{-\lambda(H-\mathbf{1})}(t) (-\lambda)(H - \mathbf{1}) \quad (3.29)$$

in view of (3.12). \square

If we interpret the i.i.d. r.v.'s J_i as waiting times between events in some point process then the sequence J_i is a renewal process and the r.v. T_n is the instant of the n -th event. The process counting the number of events occurred up to a certain time t is the counting process $N^{\bar{\nu}}(t) = \max \{n \in \mathbb{N} : T_n \leq t\}$. Clearly if one considers exponentially distributed waiting times then the corresponding counting process is the Poisson process. When the waiting times are Mittag-Leffler distributed, i.e. $P\{J > t\} = E_\alpha(\lambda t^\alpha)$, $\alpha \in (0, 1)$, $\lambda < 0$, a particular semi-Markov model on a graph have been considered in [19; 45]. The authors showed that the governing equation is time-fractional. In general if the i.i.d waiting times J_i have finite mean μ_J then one has by a simple argument using the strong law of large numbers that a.s.

$$\lim_{t \rightarrow \infty} \frac{N^{\bar{\nu}}(t)}{t} = \frac{1}{\mu_J} \quad (3.30)$$

and the elementary renewal theorem [2, Proposition 1.4] states that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} N^{\bar{\nu}}(t)}{t} = \frac{1}{\mu_J}. \quad (3.31)$$

These facts may be interpreted as an equivalence (in the long time behaviour) between the Poisson process and a general renewal process with finite-mean waiting times. Note that (3.30) and (3.31) means that if $\mathbb{E} J_i < \infty$ then as $t \rightarrow \infty$ we have, a.s., $N(t) \sim N^{\bar{\nu}}(t)$. This heuristically means that a renewal process with finite mean waiting times is indistinguishable after a ‘‘long time’’ from the Poisson process. Therefore when you observe the process $Y(t)$ defined in A3) with sojourn

times $J_n = T_{n+1} - T_n$ having finite mean then, after a transient period it behaves like the case in which $T_{n+1} - T_n$ are exponential r.v.'s.

We have here introduced a class of renewal processes associated with waiting times J such that $P\{J > t\} = q(\lambda, t)$ and under A3) we know that $\mathbb{E}J = \infty$. Therefore they never behave as a Poisson process. Corollary 3.1 and Theorem 3.2 implies that the time-changed Markov chain $X(L^f(t))$, $t \geq 0$, may be equivalently constructed starting from an embedded Markov chain Y_n and by inserting between jumps the heavy-tailed waiting times J_i .

In equation (3.6) we defined the renewal process $Y(t)$ that jumps to the state Y_n for the underlying discrete time Markov process at the arrival time T_n of the renewal process with waiting time distribution $\mathbb{P}[J > t] = q(\lambda, t)$. Then we have $Y(t) = Y_{N^\nu(t)}$, a time change using the renewal process (3.15). Next we show that the same process can also be constructed by a time change using the inverse subordinator (2.12).

Proposition 3.3. *Let $N(t)$ be a homogeneous Poisson process with rate $\theta = -\lambda$. The time-changed process $N(L^f(t))$ and the process $N^\nu(t)$ are the same process. Hence the semi-Markov process (3.6) is the same process as the time-changed Markov chain $Y_{N(L^f(t))}$.*

Proof. This is a consequence of [36, Thm 4.1] since $P\{J_i > t\} = q(\lambda, t)$ for any i and in view of Thm 2.1 it is true that $q(\lambda, t) = \mathbb{E}e^{\lambda L^f(t)}$. \square

3.1. Classification of states. We here investigate whenever a state i is transient or recurrent for the semi-Markov process $Y(t)$, by making assumptions on the embedded chain Y_n . We recall that a state i is recurrent if

$$P(\text{the set } \{t \geq 0 : Y(t) = i\} \text{ is unbounded} \mid Y(0) = i) = 1 \quad (3.32)$$

and is transient if the probability in (3.32) is zero. Since we take the probability of a tail event, 0 and 1 are the only possibilities. We have the following result.

Theorem 3.4. *Under A2), A3), and A4) it is true that*

- (1) *If the state i is recurrent for Y_n then it is recurrent for $Y(t)$*
- (2) *If the state i is transient for Y_n then it is transient for $Y(t)$*
- (3) *$\int_0^\infty q_{i,i}(t)dt = \infty$ independently from the fact that the state i is transient or recurrent.*

We recall that $q_{i,i}(t) = P\{Y(t) = i \mid Y(0) = i\} = P^i\{Y(t) = i\}$.

Proof. Note that in view of Proposition 3.3 we can write

$$Y(t) = Y_{N^\nu(t)} = Y_{N(L^f(t))}. \quad (3.33)$$

Since $t \mapsto \sigma^f(t)$ is a.s. right-continuous, unbounded, and strictly increasing, it follows from the change of variable formula in Meerschaert and Straka [39, p. 1707] that

$$\begin{aligned} \int_0^\infty \mathbb{1}_{\{t \geq 0 : Y(t) = i\}} dt &= \int_0^\infty \mathbb{1}_{\{t \geq 0 : Y_{N(L^f(t))} = i\}} dt \\ &= \int_0^\infty \mathbb{1}_{\{t \geq 0 : Y_{N(t)} = i\}} d\sigma^f(t) \\ &= \sum_n \mathbb{1}_{\{Y_n = i\}} \sum_{\tau_n \leq t < \tau_{n+1}} e(t) \end{aligned} \quad (3.34)$$

where $e(t)$ is the Poisson point process underlying the subordinator σ^f with characteristic measure $\nu(s)ds$. Note that if the state i is recurrent for Y_n then

$$P(Y_n = i \text{ for infinitely many } n) = 1 \quad (3.35)$$

and the number of summands in (3.34) is a countable infinity. Furthermore the sequence

$$\sum_{\tau_n \leq t < \tau_{n+1}} e(t) \quad (3.36)$$

is a sequence of i.i.d. r.v.'s since $e(s)$ is a Poisson point process. Therefore (3.34) is the sum of a countable infinity of i.i.d. positive r.v.'s. which diverges with probability one. This proves Item 1. If instead the state i is transient, then the number of summands in (3.34) is finite, and since our subordinators are here assumed to be not subject to killing, it is true that for all $0 \leq t_1 < t_2 < \infty$

$$0 < \sum_{t_1 \leq s \leq t_2} e(s) < \infty \text{ a.s.} \quad (3.37)$$

and the sum (3.34) is finite since it is the sum of a finite number of finite summands. This proves 2. Observe now that

$$\begin{aligned} \int_0^\infty q_{i,i}(t) dt &= \mathbb{E}^i \int_0^\infty \mathbf{1}_{\{Y(t)=i\}} dt \\ &\geq \mathbb{E}^i J_1 = \infty \end{aligned} \quad (3.38)$$

and this proves Item 3. \square

It is instructive to compare Part (3) of Theorem 3.4 with the well-known characterization of transient and recurrent states in a semi-Markov process with finite mean waiting time between jumps, using the occupation measure (or 0-potential) [15]. The semi-Markov process is formed by inserting a random waiting time J_i before jumping from state Y_{i-1} to state Y_i in the underlying Markov chain. Hence the number of times the process returns to its starting point is not affected. Hence the state is recurrent for the semi-Markov process if and only if it is recurrent for the underlying Markov chain. If the occupation times (waiting times) in each state have a finite mean, then the total expected occupation time in the starting state is proportional to the number of visits. This happens if and only if $\int_0^\infty q_{i,i}(t)dt = \infty$. However, when the waiting times between state transitions are heavy-tailed with infinite mean, the mean time spent in the starting point by the process is always infinite.

4. EXAMPLES

In this section, we provide some practical examples, to illustrate the application of the results in this paper.

Example 4.1. Consider a CTRW with iid particle jumps X_n independent of the iid waiting times W_n . Then a particle arrives at location $S(n) = X_1 + \dots + X_n$ at time $T_n = W_1 + \dots + W_n$. If $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2] < \infty$, then $n^{-1/2}S([nt]) \Rightarrow B(t)$, a Brownian motion, by Donsker's Theorem [16, Theorem 8.7.5]. If $\mathbb{P}[W_n > t] = t^{-\beta}/\Gamma(1-\beta)$ for some $0 < \beta < 1$, then $n^{-1/\beta}T_{[nt]} \Rightarrow \sigma_t^f$, a β -stable subordinator with Laplace symbol $f(\phi) = \phi^\beta$ [37, Theorem 3.41]. The number of jumps by time $t > 0$ is $N_t = \max\{n \geq 0 : T_n \leq t\}$ and a simple argument [33, Theorem 3.2] shows

that this inverse process has an inverse limit $n^{-\beta}N_{nt} \Rightarrow L_t^f$ where the inverse stable subordinator L_t^f is defined by (2.12). Then we have

$$n^{-\beta/2}S(N_{nt}) \approx (n^\beta)^{-1/2}S(n^\beta \cdot n^{-\beta}N_{nt}) \approx (n^\beta)^{-1/2}S(n^\beta L_t^f) \approx B(L_t^f)$$

as $n \rightarrow \infty$, see [33, Theorem 4.2] for a rigorous argument. The probability density $p(x, t)$ of the limit process $B(L_t^f)$ solves the time-fractional diffusion equation (2.13) with $\mathbb{C}_f(\partial_t) = \partial_t^\beta$, a Caputo fractional derivative of order $0 < \beta < 1$, and $A = D\partial_x^2$ where $D = \mathbb{E}[X_n^2]/2$, see [33, Theorem 5.1] and [37, Eq. (1.8)]. Since the Caputo fractional derivative [37, Eq. (2.16)]

$$\partial_t^\beta p(x, t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial}{\partial t} p(x, t-u) u^{-\beta} du$$

is related to the Riemann-Liouville fractional derivative [37, Eq. (2.17)]

$$\mathbb{D}_t^\beta p(x, t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_0^t p(x, t-u) u^{-\beta} du$$

by [37, Eq. (2.33)]

$$\mathbb{D}_t^\beta p(x, t) - p(x, 0) \frac{t^{-\beta}}{\Gamma(1-\beta)} = \partial_t^\beta p(x, t), \quad (4.1)$$

we can also write the governing equation (2.13) in the form

$$\mathbb{D}_t^\beta p(x, t) - p(x, 0) \frac{t^{-\beta}}{\Gamma(1-\beta)} = A p(x, t). \quad (4.2)$$

The Laplace symbol $f(\phi) = \phi^\beta$ can be computed directly from (2.1) with $a = b = 0$ and $\nu(dt) = \beta t^{-\beta-1} dt / \Gamma(1-\beta)$ using integration by parts and the definition of the Gamma function [37, Proposition 3.10]. Then $\bar{\nu}(t) = t^{-\beta} / \Gamma(1-\beta)$ and the operator \mathcal{D}^f of Toaldo [51] defined by (2.16) reduces to the Riemann-Liouville fractional derivative. Similarly, the operator \mathfrak{D}_t^f of [51] defined by (2.17) reduces to the Caputo fractional derivative. Then the governing equation (2.61) of Toaldo with $g(t) = p(x, t)$ reduces to (4.2). Since the conjugate Bernstein function $f^*(\phi) = \phi/f(\phi) = \phi^{1-\beta}$, the same calculation as for f shows that $\bar{\nu}^*(t) = t^{\beta-1} / \Gamma(\beta)$, and then the operator Φ_t of Magdziarz [31] defined by (2.21) with $M(t) = \bar{\nu}^*(t)$ is the Riemann-Liouville fractional integral

$$\mathbb{I}_t^\beta p(x, t) = \frac{1}{\Gamma(\beta)} \int_0^t p(x, t-u) u^{\beta-1} du = \frac{1}{\Gamma(\beta)} \int_0^t p(x, u) (t-u)^{\beta-1} du$$

of order β [37, p. 250]. Then the governing equation (2.62) of Magdziarz [31] with $g(t) = p(x, t)$ reduces to

$$\frac{d}{dt} p(x, t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t A p(x, s) (t-s)^{\beta-1} ds = \frac{d}{dt} A \mathbb{I}_t^\beta p(x, t). \quad (4.3)$$

Since $\frac{d}{dt} \mathbb{I}_t^\beta = \mathbb{D}_t^{1-\beta}$ we can also write (4.3) in the form

$$\frac{d}{dt} p(x, t) = \mathbb{D}_t^{1-\beta} A p(x, t) = A \mathbb{D}_t^{1-\beta} p(x, t),$$

which is commonly seen in applications [21; 41]. The heuristic derivation of (4.3) is to simply apply the operator $\mathbb{D}_t^{1-\beta}$ to both sides of (4.2), or the equivalent form $\partial_t^\beta p = A p$, but the initial condition requires some care.

Example 4.2. Replacing the iid jumps X_n in Example 4.1 with a convergent triangular array, we obtain a limit $B(L_t^f)$ where $B(t)$ is an arbitrary Lévy process [35, Theorem 3.6] with generator A given by the Lévy-Khintchine formula [32, Theorem 3.1.11]. Then all the results of Example 4.1 hold with $A = D\partial_x^2$ replaced by this Lévy generator, in \mathbb{R}^1 or in \mathbb{R}^d for any finite dimension d [35, Theorem 4.1]. The same is true more generally for Markov generators A , where the jump distribution depends on the current state [27, Theorem 4.2].

Example 4.3. For a CTRW with deterministic particle jumps $X_n = 1$, and the same waiting times W_n as in Example 4.1, we have $S(n) = n$ and the CTRW $S(N_t) = N_t$ converges to the inverse stable subordinator: $n^{-\beta}N_{nt} \Rightarrow L_t^f$. The probability density $l(x, t)$ of L_t^f solves the time-fractional equation (2.13) with $\mathbb{C}_f(\partial_t) = \partial_t^\beta$ and $A = -\partial_x$ [38, Eq. (5.7)]. Several equivalent governing equations for the inverse stable subordinator L_t^f are also discussed in [38], including equation (2.61) of Toaldo [51] (see [38, Eq. (5.9)]) and equation (2.62) of Magdziarz [31] (see [38, Eq. (5.18)]) with $g(t) = l(x, t)$ and $A = -\partial_x$. In this case, the moment generating function (2.28) can be written explicitly as $q(\lambda, t) = E_\beta(\lambda t^\beta)$ in terms of the Mittag-Leffler function

$$E_\beta(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1 + \beta j)} \quad (4.4)$$

for any $\lambda \leq 0$, see Bingham [11]. The function $q(\lambda, t)$ solves (2.26), which can be rewritten in this case as

$$\mathbb{D}_t^\beta q(\lambda, t) - \frac{t^{-\beta}}{\Gamma(1 - \beta)} = \lambda q(\lambda, t),$$

or equivalently, using (4.1), as

$$\partial_t^\beta q(\lambda, t) = \lambda q(\lambda, t).$$

That is, the moment generating function of the inverse stable subordinator is an eigenfunction of the Caputo fractional derivative. The function $q(\lambda, t)$ also solves (2.27), which can be rewritten in this case as

$$\frac{d}{dt} q(\lambda, t) = \lambda \mathbb{D}_t^{1-\beta} q(\lambda, t).$$

Here $f(\phi) = \phi^\beta$ varies regularly at zero with index β , $\bar{\nu}(t) = t^{-\beta}/\Gamma(1 - \beta)$ varies regularly at infinity with index $-\beta$, and Item (4) of Theorem 2.1 shows that $t \mapsto q(\lambda, t)$ also varies regularly at infinity with index $-\beta$, with $q(\lambda, t) \sim \lambda^{-1}t^{-\beta}/\Gamma(1 - \beta)$ as $t \rightarrow \infty$, compare Scalas [47, Eq. (24)]. The potential measure $U^{\sigma_f}(dt)$ has Laplace-Stieltjes transform $1/f(\phi) = \phi^{-\beta}$ [49, Eq. (5.12)], and inverting [37, Eq. (2.25)] shows that (2.10) holds with $c = 0$ and $u_{\sigma_f}(t) = t^{\beta-1}/\Gamma(\beta)$. Then it follows from [49, Thm 10.3] that f is a special Bernstein function. In fact $f^*(\phi) = \phi^{1-\beta}$ from (2.5) with $a^* = b^* = 0$ and $\nu^*(dt) = (1 - \beta)t^{\beta-2}dt/\Gamma(\beta)$ concentrated on $t > 0$. Note also that [49, Eq. (10.9)] $u_{\sigma_f}(t) = \bar{\nu}^*(t)$.

Example 4.4. Let $\nu(dt) = \beta t^{-\beta-1}e^{-ct}dt/\Gamma(1 - \beta)$ for some $0 < \beta < 1$ and $c > 0$ and compute

$$f(\phi) = c^\beta + \int_0^\infty (1 - e^{-\phi t}) \nu(dt) = (\phi + c)^\beta \quad (4.5)$$

using integration by parts, compare [37, Eq. (7.9)]. Then σ_t^f is a tempered stable subordinator [37, Section 7.2] killed at rate c^β : If $D(t)$ is a tempered stable subordinator with $\mathbb{E}[e^{-sD(t)}] = \exp[-t\{(\phi + c)^\beta - c^\beta\}]$ and S is an exponential random variable independent of $D(t)$ with $\mathbb{P}[S > t] = \exp[-c^\beta t]$, then we can let

$$\sigma_t^f = \begin{cases} D(t) & \text{if } t < S \\ \infty & \text{if } t > S \end{cases}$$

and it is easy to check that f is the Laplace symbol of this process. Now

$$\bar{\nu}(t) = a + \nu(t, \infty) = c^\beta + \frac{\beta}{\Gamma(1 - \beta)} \int_t^\infty s^{-\beta-1} e^{-cs} ds \quad (4.6)$$

involves the incomplete Gamma function, which cannot be written in closed form. It is well known that $\phi^\beta \tilde{u}(\phi)$ is the Laplace transform of $\mathbb{D}_t^\beta u(t)$ [37, p. 39]. Using the shift property $\tilde{u}(\phi + c) = \mathcal{L}[e^{-ct}u(t)]$ twice, it follows that the tempered fractional derivative $\mathbb{D}_t^{\beta,c}u(t) = e^{-ct}\mathbb{D}_t^\beta(e^{ct}u(t))$ has Laplace symbol f , i.e., $\mathcal{L}[\mathbb{D}_t^{\beta,c}u(t)] = (\phi + c)^\beta \tilde{u}(\phi)$, see [37, p. 209]. If u, u' are in $L^1(\mathbb{R})$ then one can write the tempered fractional derivative explicitly as [40, Theorem 2.9]

$$\mathbb{D}_+^{\beta,c}u(t) = c^\beta f(t) + \frac{\beta}{\Gamma(1 - \beta)} \int_{-\infty}^t \frac{f(t) - f(u)}{(t - u)^{\beta+1}} e^{-c(t-u)} du.$$

It follows from (2.18) that $\mathcal{D}^f u(t) = \mathbb{D}_t^{\beta,c}u(t)$, and hence the governing equation (2.61) of Toaldo can be written in the form

$$\mathbb{D}_t^{\beta,c}g(t) - \bar{\nu}(t)g(0) = Ag(t).$$

Recall that $\mathcal{L}[t^{\beta-1}/\Gamma(\beta)] = \phi^{-\beta}$, and use the shift property again, along with (2.22), to see that

$$M(t) = \mathcal{L}^{-1}[(\phi + c)^{-\beta}] = \frac{1}{\Gamma(\beta)} t^{\beta-1} e^{-ct}.$$

Then the operator Φ_t of Magdziarz and Schilling [31] defined by (2.21) can be written as

$$\Phi_t u(t) = \frac{d}{dt} \frac{1}{\Gamma(\beta)} \int_0^t u(s) M(t-s) ds = \frac{d}{dt} \mathbb{I}_t^{\beta,c} u(t)$$

where appears the tempered fractional integral [40]

$$\mathbb{I}_t^{\beta,c} u(t) := \frac{1}{\Gamma(\beta)} \int_0^t u(t-s) s^{\beta-1} e^{-cs} ds,$$

and hence the governing equation (2.62) of Magdziarz [31] reduces to

$$\frac{d}{dt} g(t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t Ag(t-s) s^{\beta-1} e^{-cs} ds = \frac{d}{dt} A \mathbb{I}_t^{\beta,c} g(t). \quad (4.7)$$

The potential measure $U^{\sigma^f}(dt)$ has Laplace-Stieltjes transform $1/f(\phi) = (\phi + c)^{-\beta}$ [49, Eq. (5.12)], and inverting shows that (2.10) holds with $c = 0$ and $u_{\sigma^f}(t) = M(t) = \bar{\nu}^*(t) = t^{\beta-1} e^{-ct} / \Gamma(\beta)$. Then it follows from [49, Thm 10.3] that f is a special Bernstein function. It is easy to check that $f(\phi) = (\phi + c)^\beta$ varies regularly at zero with index $\beta = 0$, and then Item (4) of Theorem 2.1 shows that both $\bar{\nu}(t)$ and $t \mapsto q(\lambda, t)$ are slowly varying at infinity.

Example 4.5. The (unkilled) tempered stable subordinator $D(t)$ related to the Bernstein function $(c + \phi)^\beta - c^\beta$ is also included in our framework. The Lévy density in this case is

$$\nu(t) = \frac{\beta t^{-\beta-1} e^{-ct}}{\Gamma(1-\beta)} = \frac{\beta}{\Gamma(1-\beta)} e^{-ct} \int_0^\infty e^{-st} \frac{s^\beta}{\Gamma(1+\beta)} ds. \quad (4.8)$$

Since (4.8) is the product of two completely monotone functions, it is completely monotone [49, Corollary 1.6]. This means that

$$\phi \rightarrow (\phi + c)^\beta - c^\beta = \int_0^\infty (1 - e^{-\phi t}) \frac{\beta t^{-\beta-1} e^{-ct}}{\Gamma(1-\beta)} dt \quad (4.9)$$

is a complete Bernstein function by [49, Def 6.1] and therefore it is also special by [49, Prop 7.1]. The tail of the Lévy measure here is similar to (4.6), but without the constant term:

$$\bar{\nu}(t) = \frac{\beta}{\Gamma(1-\beta)} \int_t^\infty s^{-\beta-1} e^{-cs} ds. \quad (4.10)$$

Now (2.22) becomes

$$M(t) = \mathcal{L}^{-1} \left[\left((c + \phi)^\beta - c^\beta \right)^{-1} \right] (t), \quad (4.11)$$

which seems difficult to invert in closed form. Since $a^* = 0$ in view of (2.6), we get from (2.11) that

$$\bar{\nu}^*(t) = \nu^*(t, \infty) = u_{\sigma^f}(t) \quad (4.12)$$

where $u_{\sigma^f}(t)$ is the potential density of the tempered stable subordinator. Since f is special, its conjugate $f^*(\phi) = \phi[(\phi + c)^\beta - c^\beta]^{-1}$ is the Laplace symbol of some subordinator $\sigma^*(t)$.

Example 4.6. Distributed order fractional derivatives are also included in our framework. Let $(0, 1) \ni y \rightarrow \alpha(y)$ be a function strictly between zero and one and let $p(\cdot)$ be a measure on $(0, 1)$. Choose $\alpha(y)$ and p in such a way that, for $s > 0$, it is true that

$$\begin{aligned} & \int_0^\infty (s \wedge 1) \int_0^1 \frac{\alpha(y) s^{-\alpha(y)-1}}{\Gamma(1-\alpha(y))} p(dy) ds \\ &= \int_0^1 \int_0^\infty (s \wedge 1) \frac{\alpha(y) s^{-\alpha(y)-1}}{\Gamma(1-\alpha(y))} ds p(dy) < \infty. \end{aligned} \quad (4.13)$$

Under (4.13)

$$\nu(s) = \int_0^1 \frac{\alpha(y) s^{-\alpha(y)-1}}{\Gamma(1-\alpha(y))} p(dy) \quad (4.14)$$

is a Lévy density and therefore

$$f(\phi) = \int_0^\infty (1 - e^{-s\phi}) \nu(ds) = \int_0^1 \phi^{\alpha(y)} p(dy) \quad (4.15)$$

is a Bernstein function. The operator \mathfrak{D}^f corresponding to (4.15) may be viewed as a distributed order fractional derivative [52, Remark 4.3] since here

$$\bar{\nu}(s) = \int_0^1 \frac{s^{-\alpha(y)}}{\Gamma(1-\alpha(y))} p(dy) \quad (4.16)$$

and therefore

$$\begin{aligned}
& \mathfrak{D}^f u(t) \\
&= \frac{d}{dt} \int_0^t u(s) \int_0^1 \frac{(t-s)^{-\alpha(y)}}{\Gamma(1-\alpha(y))} p(dy) ds - u(0) \int_0^1 \frac{t^{-\alpha(y)}}{\Gamma(1-\alpha(y))} p(dy) \\
&= \int_0^1 \frac{1}{\Gamma(1-\alpha(y))} \frac{d}{dt} \int_0^t u(s)(t-s)^{-\alpha(y)} ds p(dy) - u(0) \int_0^1 \frac{t^{-\alpha(y)}}{\Gamma(1-\alpha(y))} p(dy) \\
&= \int_0^1 \mathbb{D}_t^{\beta(y)} u(t) p(dy) - u(0) \int_0^1 \frac{t^{-\alpha(y)}}{\Gamma(1-\alpha(y))} p(dy). \tag{4.17}
\end{aligned}$$

The traditional form of the distributed order derivative is the special case $\alpha(y) = y$. Now note that the function $t \rightarrow t^{-\alpha(y)}$ is completely monotone for each fixed y since it is the Laplace transform of the measure $s^{\alpha(y)-1} ds / \Gamma(\alpha(y))$. Therefore, when $\alpha(y)$ and $p(\cdot)$ are such that (4.13) is fulfilled, also the function

$$\begin{aligned}
\bar{\nu}(t) &= \int_0^1 \frac{1}{\Gamma(1-\alpha(y))} \int_0^\infty e^{-ts} \frac{s^{\alpha(y)-1}}{\Gamma(\alpha(y))} ds p(dy) \\
&= \int_0^\infty e^{-ts} \int_0^1 \frac{1}{\Gamma(1-\alpha(y))} \frac{s^{\alpha(y)-1}}{\Gamma(\alpha(y))} p(dy) ds \tag{4.18}
\end{aligned}$$

is completely monotone. Therefore (4.18) is the tail of a Lévy measure with a completely monotone density by Remark 2.2. This implies that f is a complete Bernstein function [49, Def 6.1] and therefore it is also special [49, Prop. 7.1]. Hence

$$f^*(\phi) = \frac{\phi}{\int_0^1 \phi^{\alpha(y)} p(dy)} \tag{4.19}$$

is a (special) Bernstein function, and there exists a subordinator $\sigma^*(t)$ with Fourier symbol f^* . From (2.11) we have

$$\bar{\nu}^*(t) = \nu^*(t, \infty) = u_{\sigma^*}(t) \tag{4.20}$$

where $u_{\sigma^*}(t)$ is the potential density of the subordinator with Laplace exponent (4.15). Then observe that Items (1), (2) and (3) of Theorem 2.1 apply to this case. In particular Item (3) here is in accordance with [24, Thm 2.3]. Item (4) applies if p and α are such that $f(\phi) = \int_0^1 \phi^{\alpha(y)} p(dy)$ is regularly varying at $0+$. For example, if $\alpha(y) = y$ and $p(dy) = p_0(y)dy$ where p_0 is regularly varying at zero with some index $\gamma > -1$, then f is slowly varying [34, Lemma 3.1], and hence $\bar{\nu}(t)$ and $q(\lambda, t)$ are slowly varying at $t = \infty$. This is a model for ultraslow diffusion [34] where a plume of particles spreads at a logarithmic rate in time. Here the kernel of Magdziarz and Schilling [31] can be computed from

$$M(t) = \mathcal{L}^{-1} \left[\frac{1}{\int_0^1 \phi^{\alpha(y)} p(dy)} \right] (t). \tag{4.21}$$

In the special case $f(\phi) = p_1 \phi^{\beta_1} + p_2 \phi^{\beta_2}$ (retarding subdiffusion) one can write

$$M(t) = p_2^{-1} t^{\beta_2} E_{\beta_2 - \beta_2, \beta_2 + 1} \left(-\frac{p_1}{p_2} t^{\beta_2 - \beta_1} \right),$$

where the two parameter Mittag-Leffler function

$$E_{s,t}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(sn + t)},$$

compare Chechkin et al. [14, Eq. (16)].

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